2D Euclidean Quantum Gravity

Weil–Petersson Volumes and Intersection Theory on Moduli Space

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Abstract

We review recent progress in formulating and solving the quantum theory of Euclidean Einstein-Hilbert gravity in two dimensions. While the classical theory is topological and has no dynamics, its quantum counterpart is a highly nontrivial topological field theory. The main characters in our story are the moduli spaces of Riemann surfaces, which parametrize gauge equivalence classes of metrics on two-dimensional surfaces. The study of their geometric structure—in particular, their symplectic forms, volumes, and intersection numbers—gives us the tools to understand and calculate the partition function and correlators of the theory.

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1 Introduction and Motivation

When I was younger and less burdened by my lack of knowledge of physics, I thought to myself: "come on—how hard can quantum gravity be?" I had just learned that gravity could be formally quantized by doing a path integral over all configurations of the spacetime metric field, so I decided to give it a try. I chose to work in Euclidean signature, where the math seemed to make more sense: spacetimes are just Riemannian manifolds, and the path integral has no imaginary numbers in it and generally has better convergence properties.

I started with the case of dimension two: dimension zero is trivial, and dimension one has no gravity at all, since all curvature tensors vanish identically there. My attempt to write down the partition function Z of the theory started like this:

$$Z = \int \mathcal{D}g \, e^{-S_{\mathrm{E}}[g]}, \qquad S_{\mathrm{E}}[g] = -\frac{1}{16\pi G} \int_M \mathrm{d}^2x \sqrt{g} \, R. \tag{1.1}$$

Here g is the metric on a 2-manifold M, R is its scalar curvature, and S_{E} is the corresponding Euclidean Einstein-Hilbert action. By the Gauss-Bonnet theorem, S_{E} is topological:

$$S_{\mathrm{E}} = -\frac{1}{16\pi G} \int_M \mathrm{d}^2x \sqrt{g} \, R = -\frac{1}{16\pi G} \cdot 4\pi\chi(M) = -\frac{\chi(M)}{4G}. \tag{1.2}$$

For a Riemannian 2-manifold without boundary, $\chi(M) = 2 - 2h$ can be expressed in terms of the genus h of M. We may therefore calculate the partition function of 2D gravity as a sum, over all genera $h \in \mathbb{N}$, of path integrals over metrics g_h of a fixed genus h:

$$Z = \int \mathcal{D}g \, e^{-S_{\mathrm{E}}[g]} = \int \mathcal{D}g \, e^{\frac{2-2h}{4G}} = \sum_{h=0}^{\infty} \int \mathcal{D}g_h \, e^{\frac{1-h}{2G}} = e^{-\frac{1}{2G}} \sum_{h=0}^{\infty} e^{-\frac{h}{2G}} (\mathrm{vol}_h). \tag{1.3}$$

This is where I got stuck: the topological nature of the action makes the path-integrand constant, so I was left with the problem of computing the volumes of some vaguely-defined spaces of metrics on 2-dimensional surfaces. I didn't know how to proceed.

The purpose of this review is to explain how to define and compute these volumes. The story, due primarily to Maryam Mirzakhani and Edward Witten, is astonishingly beautiful: moduli spaces of Riemann surfaces are introduced as natural candidates for the spaces of metrics I bemoaned above, their volumes are computed recursively, and a powerful method is developed for extracting the correlation functions of topological gravity directly from these volumes. The tale weaves its way through Riemann surfaces, symplectic geometry, topological recursion, and intersection theory, and culminates with a surprising connection to the theory of random matrices. For sanity's sake, we will leave out the matrix part of the story, focusing primarily on its geometrical aspects.

This paper is organized in two main parts. In the first part, we define moduli spaces of Riemann surfaces and review Mirzakhani's computation of their volumes. In the second part, we develop intersection theory on these spaces and explain Witten's approach to correlators in topological gravity. Our discussion and commentary is interspersed throughout the paper, and our short conclusion provides a quick summary.

2 Moduli of Riemann Surfaces

What is the space $\mathcal{M}$ of metrics on which to define the path integral (1.1)? The most naïve answer is simply that $\mathcal{M}$ should be the set of all Riemannian metrics on any 2D surface. But this set is much too large to be useful: the action S_{E} is constant on large swaths of this set, and this makes Z formally infinite. It behooves us to organize the set of metrics into equivalence classes on which S_{E} is constant, pick a single representative of each equivalence class, and integrate $e^{-S_{\mathrm{E}}}$ only over the set of representatives. In other words, we must quotient out by the symmetries of S_{E}.

Physicists often say that the Einstein-Hilbert action is "diffeomorphism invariant." What they mean mathematically is that S_{E} assumes the same value on any two metrics g, g' if the associated Riemannian manifolds (M, g) and (M', g') are isometric [1, 2]. So our first attempt will be to take $\mathcal{M}$ to be the set of isometry classes of metrics on a 2D surface. Even then, however, $\mathcal{M}$ is too large. The action has another symmetry: it is invariant under constant global rescalings of the metric $g \longrightarrow g' = \alpha g$, with $\alpha \in \mathbb{R}_{+}$. This is because the scaling of the curvature R cancels the scaling of the volume form $\mathrm{d}^2x\sqrt{g}$ in the integral defining S_{E}. In fact, this observation and its proof generalize[1] to *local* scale transformations of the form $g \longrightarrow g' = ug$, where u is any positive function on the original manifold. These are variously called *conformal* or *Weyl* transformations, and we must quotient them out as well. Thus we arrive at a second attempt: $\mathcal{M}$ should be the set of metrics on a 2D surface up to isometry and scale. Points in this set are called *conformal structures*.

This is almost a satisfactory definition. Before proceeding, we will impose one more restriction: the surfaces we consider must be orientable.[2] With this restriction, a miracle occurs: every conformal structure on a surface automatically gives rise to a *complex structure* on the surface which turns it into a *Riemann surface*:

Definition 2.1. A *Riemann surface* is a connected complex manifold X of complex dimension one. Equivalently, a Riemann surface is an oriented manifold of (real) dimension two, together with a conformal structure.

We conclude that the space $\mathcal{M}$ over which to perform the path integral in (1.1) is the set of conformal structures on Riemann surfaces. It is called the *moduli space of Riemann surfaces*, and we will soon discover that it carries an enormous amount of hidden structure.

2.1 Uniformization and Moduli Spaces

Having defined Riemann surfaces, our first goal is to characterize and classify them. By way of taxonomy, they look like rubber sheets, possibly with holes, boundaries, or punctures. To begin, we will dispense with boundaries and punctures—we will return to them shortly. In this simplified setting, our first result towards a classification of these surfaces is the *Riemann uniformization theorem*, which says that the topology (in particular, the genus[3]) of a Riemann surface completely determines its geometry [3].

[1]This follows because the Jacobian of the transformation cancels out at each point.

[2]This restriction is not strictly necessary, but without it things become more difficult down the line. One may regard this choice as an explicit specification of the quantum theory whose physics we wish to study.

[3]We will henceforth change notation and use g instead of h to refer to the genus.

Theorem 2.2 (Riemann uniformization)**.** *Every Riemann surface X has universal cover $\widetilde{X}$ conformally equivalent to the Riemann sphere $\mathbb{P}^1$, the complex plane $\mathbb{C}$, or the disk $D^2 \subset \mathbb{C}$.*

- *If $\widetilde{X} \simeq \mathbb{P}^1$, then X has constant curvature $K = +1$, genus $g = 0$, and trivial fundamental group. That is, $X \simeq S^2 = \Sigma_0$ (an apple) is the Riemann sphere.*
- *If $\widetilde{X} \simeq \mathbb{C}$, then X has constant curvature $K = 0$, genus $g = 1$, and fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. That is, $X \simeq T^2 = \Sigma_1$ (a donut) is a torus.*
- *If $\widetilde{X} \simeq D^2$, then X has constant curvature $K = -1$, genus $g > 1$, and nonabelian fundamental group. That is, $X \simeq \Sigma_g$ (a pringle) is a hyperbolic surface.*

The uniformization theorem gives us license to refer to Riemann surfaces by their genus, so we will write $X = \Sigma_g$. It also splits the full moduli space $\mathcal{M}$ into the disjoint union of the moduli spaces $\mathcal{M}_g$ of Riemann surfaces of genus g. We now seek a description of $\mathcal{M}_g$: that is, we want to know how many conformal structures exist on a surface of fixed genus. To attack this question, we observe that $\mathcal{M}_g$ is the set of all Riemann surfaces up to isometry.[4] Thus we write $\mathcal{M}_g = \widetilde{\mathcal{M}}_g/\mathrm{Diff}(\Sigma_g)$, where $\widetilde{\mathcal{M}}_g$ is the set of *all* Riemann surfaces of genus g, and $\mathrm{Diff}(\Sigma_g)$ is the group of all Riemann surfaces to which Σ_g is isometric.[5] Unfortunately, $\mathrm{Diff}(\Sigma_g)$ is usually very hard to describe, so we proceed in two steps [3]:

1. Study the *Teichmüller space*[6] $\mathcal{T}_g = \widetilde{\mathcal{M}}_g/\mathrm{Diff}_0(\Sigma_g)$, where $\mathrm{Diff}_0(\Sigma_g)$ is the connected component of the identity in $\mathrm{Diff}(\Sigma_g)$. That is, instead of identifying *all* isometric surfaces, we only identify Σ_g and Σ_g' if all pairs of isometries between them are homotopic. In less intimidating words, we identify all surfaces that are isometric to Σ_g by a diffeomorphism sufficiently close (i.e. continuously deformable) to the identity map.
2. Finish the job by dividing out the *mapping class group* (MCG) $\Gamma_g = \mathrm{Diff}(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$ to obtain the moduli space $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$. The MCG contains (homotopy classes of) "large gauge transformations," or isometries not continuously nonnected to the identity. Usually Γ_g is finite or discrete; it captures what makes $\mathcal{M}_g$ topologically nontrivial.

It turns out that $\mathrm{Diff}_0(\Sigma_g)$ contains "most" of the isometries of Σ_g, in that the quotient space $\mathcal{T}_g$ is finite-dimensional and relatively easy to describe. However, $\mathcal{T}_g$ usually has infinite volume, and its quotient by Γ_g fixes this problem by making $\mathcal{M}_g$ compact. We will now put this procedure to work and describe the spaces $\mathcal{T}_g$ and $\mathcal{M}_g$ for all genera, in the process making the notions in this paragraph more precise. We shall discover that the moduli space is trivial in genus zero, rather tame in genus one, and extremely interesting in $g > 1$.

[4] Contrast this description with our attempt to define $\mathcal{M}$ as the set of 2-manifolds up to isometry and scale. By requiring our 2-manifolds to be Riemann surfaces, uniformization makes the phrase "up to isometry" sufficient to describe all conformal structures on a surface of fixed genus non-redundantly.

[5] Many physicists say for shorthand that $\mathrm{Diff}(\Sigma_g)$ is the "diffeomorphism group" of Σ_g. This abuse of terminilogy is justified, as long as it is clear that we refer only to the *active* diffeomorphisms of Σ_g, which pull back the metric of Σ_g under the coordinate transformation that defines the diffeomorphism.

[6] Teichmüller was an ardent Nazi, and we will not allow his name to appear in this paper any more than necessary. Interestingly, the extension of his work described here is due to an Iranian woman and a Jew.

2.2 Classification of Surfaces

Genus zero. It is a standard result that the Riemann sphere $\mathbb{P}^1$ is the *only* compact Riemann surface in genus $g = 0$. That is, every genus-zero surface is conformally equivalent to $\mathbb{P}^1$. Therefore in this case the spaces $\mathcal{T}_0 = \mathcal{M}_0 = \{\mathbb{P}^1\}$ both consist of a single point.

Genus one. Every torus can be constructed identifying the opposite sides of a parallelogram. In more precise language, every torus can be realized as the quotient of the complex plane $\mathbb{C}$ by a nondegenerate lattice $\Lambda = \tau_1\mathbb{Z} \oplus \tau_2\mathbb{Z} \subset \mathbb{C}$. One may then ask which linearly independent generators $\tau_1, \tau_2 \in \mathbb{C}$ yield inequivalent conformal structures. Towards an answer, note that any two lattices (i.e. any two parallelograms) related to each other by rotations or dilations give rise to conformally equivalent tori. Thus without loss of generality, we may twist and scale the lattice to set $\tau_1 = 1$. Next, observe that the lattice generated by 1 and τ_2 is a reflection of the one generated by 1 and $\bar{\tau}_2$ about the real axis. These lattices evidently yield equivalent tori, so without loss of generality we may take $\tau_2 \equiv \tau \in \mathbb{H}_+$. We conclude that the *modulus* τ parametrizes $\mathcal{T}_1$, so we have $\mathcal{T}_1 = \mathbb{H}_+$. There follows a beautiful story about Möbius transformations and modular invariance. To keep the length of this paper finite, we omit the details. The end result [4] is that $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$, so that $\mathcal{M}_1 = \mathbb{H}_+/\mathrm{PSL}(2,\mathbb{Z})$. In fact, it turns out that $\mathcal{M}_1$ is compact and inherits the natural hyperbolic metric from $\mathbb{H}_+$. Its volume can be computed directly and is exactly $\frac{\pi}{3}$.

Genus two and higher. Riemann surfaces combine three types of structure: conformal, complex, and Riemannian. We took the conformal viewpoint in genus zero and the complex viewpoint in genus one; here the metric viewpoint is useful. Each isometry class of a genus-g surface[7] is called a *hyperbolic structure*. We will describe the space of such structures by carving Σ_g into simpler pieces called *pairs of pants* (surfaces homeomorphic to a 3-holed sphere), counting hyperbolic structures on the pants, and then gluing the pants together.

By an inductive argument, it may be shown [5] that any genus-g surface can be cut along $3g - 3$ simple closed curves γ_i into a disjoint union of $2g - 2 = -\chi(\Sigma_g)$ pairs of pants. (One decomposition for $g = 2$ is shown in Fig. 1.) Pants decompositions are not unique, but every decomposition uses $3g - 3$ curves and produces $2g - 2$ pants. Each pair of pants inherits a hyperbolic metric from the surface whence it was hewn, so it remains to describe the space of hyperbolic structures on a single pair of pants, and then to explain how to glue pairs of pants back together in different ways to produce inequivalent hyperbolic structures on Σ_g.

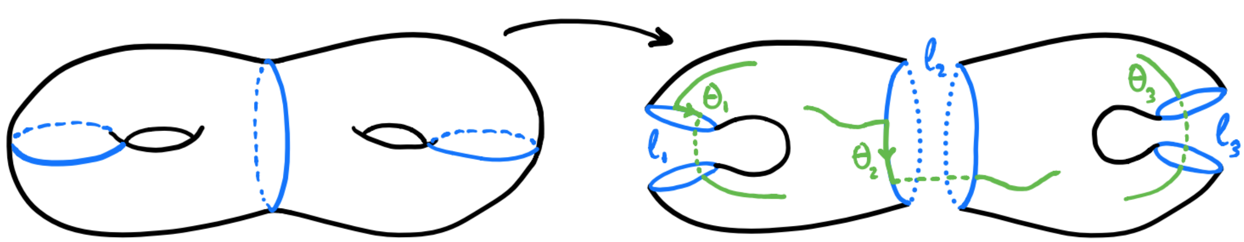

Figure 1: A pants decomposition of Σ_2, illustrating the length and twist parameters.

[7]From now on, we take $g > 1$ unless explicitly stated, so that Σ_g is a hyperbolic surface by uniformization.

In fact, any Riemann surface admits a pants decomposition by totally geodesic curves. This fact proves extremely useful in light of the following result [6]:

Proposition 2.3. *Every pair of pants with totally geodesic boundary components of fixed lengths* $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3) \in \mathbb{R}_+^3$ *admits a unique hyperbolic structure.*

So the space $\mathcal{T}_g$ (not yet $\mathcal{M}_g$) is the set of inequivalent hyperbolic structures obtained by gluing together pairs of pants along $3g-3$ curves of specified lengths.[8] Varying any of these *length parameters* $\ell_i \in \mathbb{R}_+$ changes the metric and hyperbolic structure of Σ_g. In addition, there are also $3g-3$ *twist parameters* $\theta_i \in \mathbb{R}$ (illustrated below) that describe how the boundary components of the pants are screwed onto each other. Thurston [7] writes:

> "That a twist parameter takes values in $\mathbb{R}$, rather than S^1, tends to be a confusing issue... But, remember, to determine a point in $\mathcal{T}_g$ we need to consider how many times the leg of the pajama suit is twisted before it fits onto the baby's foot."

In any case, the $6g-6$ length and twist parameters are known as *Fenchel-Nielsen* (FN) *coordinates* on $\mathcal{T}_g = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$, which thereby becomes a smooth manifold. In fact, more is true [8]: $\mathcal{T}_g$ has the structure of a phase space, and the coordinates $(\ell_i, \theta_i) \sim (q_i, p_i)$ are canonically conjugate to each other in the sense of Hamiltonian mechanics.

Theorem 2.4. *The space* $\mathcal{T}_g = \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$ *is a* $(6g-6)$*-dimensional symplectic manifold. Its symplectic form is given in Fenchel-Nielsen coordinates by*

$$\omega \equiv \sum_{i=1}^{3g-3} \mathrm{d}\ell_i \wedge \mathrm{d}\theta_i. \tag{2.1}$$

The existence of this symplectic form, aside from being completely miraculous, makes the object ω^{3g-3} a volume form on $\mathcal{T}_g$. Unfortunately, the volume of $\mathcal{T}_g$ is infinite. In a sense that we will make precise below, dividing out Γ_g causes the quotients $\mathcal{M}_g$ to have finite volume. These volumes were computed recursively by Maryam Mirzakhani. But before we describe her work, we need to outfit our construction with boundaries.

2.3 Weil–Petersson Volumes

Suppose, therefore, that our Riemann surface has n boundaries in addition to g holes. Without loss of generality, we may take the boundaries to be geodesic circles of lengths $\mathbf{L} = L_1, ..., L_n$. If any of the L_i vanishes, then the corresponding boundary degenerates into a puncture. The Euler characteristic of such a surface is $2-2g-n$, and for $g > 1$, the space $\mathcal{T}_{g,n}(\mathbf{L})$ parametrizes hyperbolic surfaces of genus g with n boundaries of fixed length $\mathbf{L}$ and has dimension $6g-6+2n$.[9] Moreover, $\mathcal{T}_{g,n}$ retains a symplectic structure.

[8] There are $2g-2$ pairs of pants, each with 3 boundary curves, for a total of $6g-6$ boundary circles. But half of these curves must be identified in stitching Σ_g back together, for a total of $3g-3$ length parameters.

[9] The pants decomposition now cuts the surface along $3g-3+n$ curves, each of which contributes a length and a twist parameter for a total of $6g-6+2n$ FN coordinates. Crucially, the n boundary curves have fixed lengths $\mathbf{L}$ and are not glued to anything, so they contribute neither length nor twist parameters.

Next, we quotient by $\Gamma_{g,n}(\mathbf{L})$, which can be rather complicated. We then obtain $\mathcal{M}_{g,n}(\mathbf{L})$, but in a nasty surprise the moduli space is non-compact! Fortunately, the *Deligne-Mumford compactification* $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ is available to "plug the holes" in $\mathcal{M}_{g,n}(\mathbf{L})$ by including degenerate surfaces like those with pinches. With these upgrades, we state a stunning theorem [8]:

Theorem 2.5. $\overline{\mathcal{M}}_{g,n}(\mathbf{L}) \sim \mathbb{R}^{6g-6+2n}$ *has the following topological and geometric properties:*

- *It is simply connected and compact. When* $\mathbf{L} = \mathbf{0}$ *(i.e. if only punctures are allowed), it is also orientable. Topologically, we have* $\overline{\mathcal{M}}_{g,n}(\mathbf{L}) = \overline{\mathcal{M}}_{g,n}(\mathbf{0}) \equiv \overline{\mathcal{M}}_{g,n}$.
- *The quotient by* $\Gamma_{g,n}$ *makes it an orbifold.*[10] *In particular,* $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ *admits a finite cover by a manifold. It looks like a manifold with a finite number of folds and corners.*
- *It has a real analytic structure, and when* $\mathbf{L} = 0$*, this is also a complex structure.*
- *It inherits the natural Riemannian metric and symplectic structure from* $\mathcal{T}_{g,n}(\mathbf{L})$*. In Fenchel-Nielsen coordinates, the Weil–Petersson symplectic form on* $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ *is*

$$\omega_{\mathbf{L}} = \sum_{i=1}^{3g-3+n} \mathrm{d}\ell_i \wedge \mathrm{d}\theta_i. \tag{2.2}$$

The last of these results, proven by Wolpert (and sometimes called "Wolpert's magic formula"), assures us that $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ has a volume form, and by compactness this volume is finite. Since $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ is topologically equivalent to $\overline{\mathcal{M}}_{g,n}(\mathbf{0})$, one may ask if their symplectic structures (and hence volumes) are equal. The answer is no, and the precise relation between the symplectic forms $\omega_{\mathbf{L}}$ and $\omega_{\mathbf{0}}$ is the content of the *Duistermaat–Heckman* (DH) *theorem*, which we will soon build up enough technology to state. In any case, we are led naturally to define the volumes of the spaces $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ using their symplectic forms. Our definition comes with some conventional prefactors [9], placed with a great deal of foresight.

Definition 2.6. The *Weil–Petersson* (WP) *volumes* of the moduli spaces $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ are

$$V_{g,n}(\mathbf{L}) \equiv \frac{1}{(2\pi^2)^{3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}(\mathbf{L})} \frac{\omega_{\mathbf{L}}^{3g-3+n}}{(3g-3+n)!} \text{ `` = '' } \frac{1}{(2\pi^2)^{3g-3+n}} \int_{\overline{\mathcal{M}}_{g,n}(\mathbf{L})} e^{\omega_{\mathbf{L}}}. \tag{2.3}$$

The last "equality" makes sense if one treats $e^{\omega_{\mathbf{L}}}$ as a power series in $\omega_{\mathbf{L}}$ and observes that only the top power of $\omega_{\mathbf{L}}$ can be integrated over $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$, so all other terms vanish. Despite appearances to the contrary, the WP volumes can actually be computed. For example, we will soon be able to prove (e.g.) that $V_{1,1}(\mathbf{0}) = \frac{1}{2\pi^2}\left(\frac{L^2}{24} + \frac{\pi^2}{6}\right)$. In the next subsection, we will loosely describe Mirzakhani's approach to computing these volumes using a recursive argument based on pants. The main result of the analysis is that $V_{g,n}(\mathbf{L})$ is a polynomial in the L_i^2 whose coefficients are positive rational multiples of powers of π.

N.B. Our exposition below primarily follows [9], but is heavily influenced by several of Mirzakhani's beautiful papers on the geometry of moduli spaces: [10, 11, 12, 13].

[10]To be precise and/or unbearably pedantic, $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ is actually an algebro-geometric generalization of an orbifold called a *stack*; in the literature, it is often called the moduli stack of stable curves.

2.4 Topological Recursion

We have already introduced the idea of cutting a Riemann surface along a closed geodesic to split it into components with fewer moduli. Mirzakhani's idea was that given the volumes of these simpler moduli spaces, one can integrate these volumes over the choice of cutting curve to obtain the volume of the original moduli space.

Of course, every surface eventually falls apart into pairs of pants, but we imagine making the cuts one by one. We focus on the result of the first cut: it can either split the surface into two, in which case we have cut along a *separating curve*; or not, in which case we have cut along a *non-separating curve*. In either case, we take the curve to be a simple, closed geodesic of length b. To describe what happens in each case (see Fig. 2), we denote by $\Sigma_{g,n}(\mathbf{L})$ a Riemann surface of genus g with n geodesic boundaries of lengths $\mathbf{L} = (L_1, ..., L_n) \in \mathbb{R}^n_+$. We will assume for the moment that the original surface has no boundaries.

- If the curve is separating, then it splits Σ_g into two surfaces of genera $g_1, g_2 < g$, with $g_1 + g_2 = g$. It also introduces a boundary on each one: $\Sigma_g \rightsquigarrow \Sigma_{g_1,1}(b) \sqcup \Sigma_{g_2,1}(b)$.
- If the curve is non-separating, then it opens up one of the holes on Σ_g. It thus lowers the genus by 1; it also introduces two equal boundaries: $\Sigma_g \rightsquigarrow \Sigma_{g-1,2}(b, b)$.

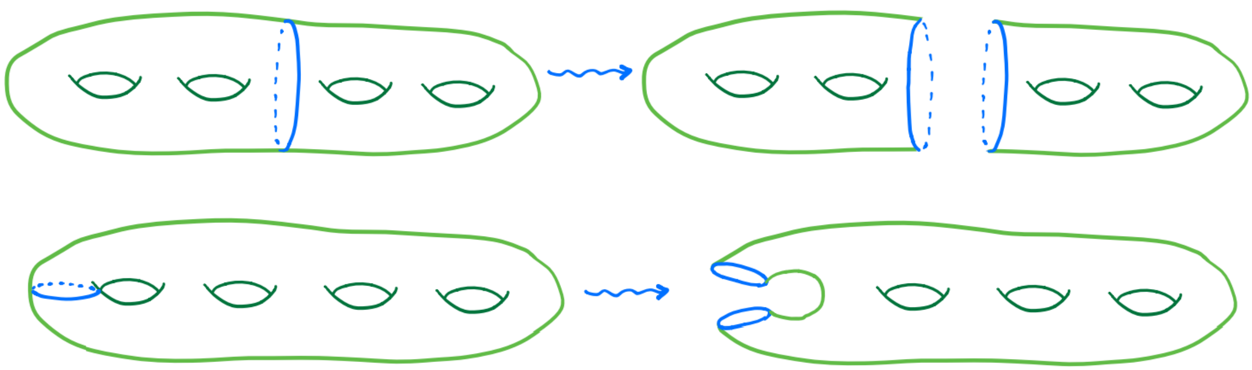

Figure 2: Separating and non-separating geodesics and the moduli they introduce.

If all of the lower-genus WP volumes $V_{g',n}$ (for $g' < g$) are known,[11] then one might hope to obtain a recursive formula for V_g by first counting up the number of hyperbolic structures on the simpler surface(s), and then integrating over all possible choices of the boundary length $b \in \mathbb{R}_+$ and its associated twist parameter $\theta \in [0, b)$.[12] That is, we hope for

$$V_g \stackrel{?}{=} \begin{cases} \displaystyle\int_0^\infty \mathrm{d}b \int_0^b \mathrm{d}\theta \, V_{g_1,1}(b) V_{g_2,1}(b) & \text{(separating)}, \\ \displaystyle\int_0^b \mathrm{d}b \int_0^b \mathrm{d}\theta \, V_{g-1,2}(b, b) & \text{(non-separating)}. \end{cases} \tag{2.4}$$

[11] More precisely, we only need to know $V_{g',1}$ for all $1 < g' < g$ and $V_{g-1,2}$.

[12] Notice that $\theta \in S^1(b)$ is valued in a circle, not in $\mathbb{R}$, because we are working directly in the moduli space.

Finally, suppose that the original surface comes to us with boundaries of its own—say n of them, with prescribed lengths $\mathbf{L}$. In the separating case, these boundaries will be partitioned arbitrarily among the two components created by the cut. And in the non-separating case, the two new boundaries (b, b) will be appended to the list $\mathbf{L}$ of pre-existing boundaries.

The main problem with this idea is that there is no canonical choice of geodesic along which to cut. Mirzakhani dealt with this problem by choosing *all* of them. To wit: let $\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})$ denote the set of all pairs (X, γ), where X is a hyperbolic structure and γ is a simple closed geodesic on X. One should view $\pi\colon \overline{\mathcal{M}}^*_{g,n}(\mathbf{L}) \longrightarrow \overline{\mathcal{M}}_{g,n}(\mathbf{L})$ as a bundle, where the fiber over $X \in \overline{\mathcal{M}}_{g,n}(\mathbf{L})$ is the set S_X of simple closed geodesics on X, and the projection $\pi\colon (X, \gamma) \longmapsto X$ forgets γ. Suppose that we had a function $F\colon \mathbb{R}_+ \longrightarrow \mathbb{R}$ that acts somewhat like a measure on the set of lengths b_γ of geodesics $\gamma \in S_X$, in the sense that F satisfies

$$\sum_{\gamma \in S_X} F(b_\gamma) = \int\limits_{\pi^{-1}(X)} F(b_\gamma) = 1. \tag{2.5}$$

Then one can find the volume of $\overline{\mathcal{M}}_{g,n}(\mathbf{L})$ by "unfolding the integral" and integrating against F over the total space $\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})$. Denoting the WP volume form by ν, we have

$$V_{g,n}(\mathbf{L}) = \int\limits_{\overline{\mathcal{M}}_{g,n}(\mathbf{L})} \nu = \int\limits_{\overline{\mathcal{M}}_{g,n}(\mathbf{L})} 1 \cdot \nu = \int\limits_{\overline{\mathcal{M}}_{g,n}(\mathbf{L})} \left(\sum_{\gamma \in S_X} F(b_\gamma) \right) \nu = \int\limits_{\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})} F(b_\gamma) \nu. \tag{2.6}$$

In the last step, we realized that the sum over S_X was being performed once for every surface $X \in \overline{\mathcal{M}}_{g,n}(\mathbf{L})$, so the notation could be cleaned up by integrating over $\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})$ instead.

The view from $\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})$ clarifies the discussion above. The proposal to integrate over b and θ was a dimly realized attempt to parametrize $\overline{\mathcal{M}}^*_{g,n}(\mathbf{L})$ by the length and twist parameters of $\gamma \in S_X$. (One imagines cutting along γ and then gluing X back together; changing either the length or twist of γ affects both the hyperbolic structure of X and the curve γ itself.) This parametrization is only really correct when a single cut along γ instantly decomposes X into pairs of pants. In the general case, one must develop the more sophisticated machinery of *multi-curves* to slice X into multiple pants at once [8]. We will not attempt to do so here. Nevertheless, the main ideas of Mirzakhani's recursion are captured in the following formula [9], which summarizes our discussion so far:

$$V_{g,n}(\mathbf{L}) \sim \int_0^\infty \mathrm{d}b\, b F(b) \left[\frac{1}{2} \sum_{\substack{g_1, g_2 \\ g_1 + g_2 = g}} \sum_{\substack{\mathbf{L}_1, \mathbf{L}_2 \\ (\mathbf{L}_1, \mathbf{L}_2) = \mathbf{L}}} V_{g_1, n_1+1}(\mathbf{L}_1, b) V_{g_2, n_2+1}(\mathbf{L}_2, b) + V_{g-1, n+2}(\mathbf{L}, b, b) \right]. \tag{2.7}$$

This formula improves (2.4) by including the magic function F in the integrand and performing the trivial θ integral to get a factor of b. In the first term, which describes separating curves, we sum over all partitions of genera and boundaries, and the factor of $\frac{1}{2}$ accounts for overcounting if the two sub-surfaces cut by γ are exchanged. The second term describes the non-separating case and is straightforward. All together, this formula captures the spirit, if not all of the details, of topological recursion à la Mirzakhani.

2.5 The Calculation of $V_{1,1}$

The abstract ideas introduced above can be made very explicit in the case of $\overline{\mathcal{M}}_{1,1}(0)$, the (compactified) moduli space of punctured hyperbolic tori. The calculation begins by taking note of the miraculous *McShane identity* [14]:

Proposition 2.7 (McShane). *Let $X \in \mathcal{M}_{1,1}(0)$ by a hyperbolic torus with one cusp. Then,*

$$\sum_{\gamma \in S_X} \frac{1}{1+e^{\ell_\gamma}} = \frac{1}{2}. \tag{2.8}$$

The sum runs over all simple, closed geodesics γ on X with length b_γ.

This identity pulls the magic function $F(b_\gamma) = 2/(1+e^{\ell_\gamma})$ out of a hat. With some more work, however, Mirzakhani was able to find a family of such functions for all other (g,n). With the McShane identity in hand, Mirzakhani views the sum over S_X as an integration along the fiber of $\overline{\mathcal{M}}^*_{1,1}(0)$ at X. She thereby unfolds the integral:

$$V_{1,1}(0) \;=\; \int\limits_{\overline{\mathcal{M}}_{1,1}(0)} \nu \;=\; \int\limits_{\overline{\mathcal{M}}_{1,1}(0)} 1\cdot\nu \;=\; \int\limits_{\overline{\mathcal{M}}_{1,1}(0)} \left(\sum_{\gamma\in S_X} \frac{2}{1+e^{\ell_\gamma}}\right)\nu \;=\; \int\limits_{\overline{\mathcal{M}}^*_{1,1}(0)} \left(\frac{2}{1+e^{\ell_\gamma}}\right)\nu. \tag{2.9}$$

The next step is to give an explicit parametrization of $\overline{\mathcal{M}}_{1,1}(0)$ by Fenchel-Nielsen coordinates, i.e. length and twist parameters [8]. To do so, we cut each punctured torus $X \in \overline{\mathcal{M}}_{1,1}(0)$ along a geodesic $\gamma \in S_X$, and then glue X back together. Happily, the topology of once-punctured tori is simple enough that *every* such cut opens up the hole in X, as seen in Fig. 3. That is, each cut produces a degenerate pair of pants whose "waist" is the puncture on X.[13] By the uniqueness of hyperbolic structures on pants, we conclude that each pair $(X,\gamma) \in \overline{\mathcal{M}}^*_{1,1}(0)$ is labeled uniquely by a single length parameter—the length $\ell_\gamma \in \mathbb{R}_+$ of γ—and a single twist parameter $\theta_\gamma \in [0, b_\gamma)$. Crucially, this parametrization was not available for $\overline{\mathcal{M}}_{1,1}(0)$, which did not come with geodesics to cut along. In any case, we coordinatize $\overline{\mathcal{M}}^*_{1,1}(0)$ as a triangular wedge in $\mathbb{R}^2$:

$$\overline{\mathcal{M}}^*_{1,1}(0) = \left\{(\ell,\theta)\in\mathbb{R}^2 \,\middle|\, \ell>0,\ \theta\in[0,\ell)\right\}. \tag{2.10}$$

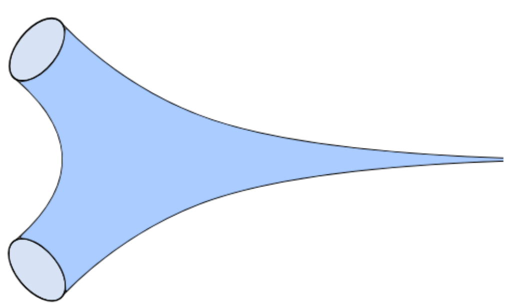

Figure 3: A degenerate pair of pants that, perhaps, doubles as a subtle critique by Mirzakhani of the unrealistic aesthetic expectations placed on the body by modern society.

[13]The geodesics considered here must be closed, so they cannot hit the puncture on X.

The Weil-Petersson form is $\omega = \mathrm{d}\ell \wedge \mathrm{d}\theta$, so we have $\nu = \frac{1}{2\pi^2}(\mathrm{d}\ell \wedge \mathrm{d}\theta)$, and therefore

$$V_{1,1}(0) = \int_{\overline{\mathcal{M}}^*_{1,1}(0)} \left(\frac{2}{1+e^{\ell_\gamma}}\right)\nu = \frac{1}{2\pi^2}\int_0^\infty \int_0^\ell \left(\frac{2}{1+e^{\ell}}\right)\mathrm{d}\ell\,\mathrm{d}\theta = \frac{1}{2\pi^2}\int_0^\infty \frac{2\ell}{1+e^\ell} = \frac{1}{12}. \quad (2.11)$$

Two final remarks are in order. First of all, the moduli space $\mathcal{M}_{1,1}(0)$ is special in that every point is a so-called *orbifold point*, i.e. a corner or fold of what used to be a smooth manifold. This strange happenstance comes from the existence an extra $\mathbb{Z}_2$ symmetry of the surfaces X in the case $(g,n) = (1,1)$. Due to the symmetry, we must further divide $V_{1,1}$ by two [3]:

$$V_{1,1}(0) = \frac{1}{24}. \quad (2.12)$$

Second of all, the result above is usually reported as $V_{1,1} = \frac{\pi^2}{12}$. The discrepancy is due to our normalization of the Weil-Petersson form by the extra conventional factor of $2\pi^2$.

3 Intersection Theory on Moduli Space

To be continued...

3.1 Intersection Numbers

3.2 Yang-Mills vs. Gravity

3.3 The MMM Classes

3.4 Correlators at Last

3.5 Comments on Topological Gravity

4 Summary and Conclusions

In this review, we have explored some properties of the moduli spaces of Riemann surfaces. Our story began with some basic musings on Euclidean quantum gravity in 2 dimensions, where these moduli spaces appeared as the natural domains of integration for the gravitational path integral. We classified these spaces using the Riemann uniformization theorem and quickly focused on the case of hyperbolic surfaces, whose moduli spaces $\mathcal{M}_{g,n}$ are extremely rich. Using geometric constructions like pants decompositions, we found that they admit orbifold compactifications $\overline{\mathcal{M}}_{g,n}$ which carry a natural Weil-Petersson symplectic form. This symplectic structure allowed us to recount a recursive method, developed by Mirzakhani, for computing the volumes of the moduli spaces.

From a naïve perspective, the WP volumes allow one to directly obtain the numerical value of the partition function of 2D Euclidean Einstein-Hilbert gravity. It would be interesting, out of pure curiosity, to actually carry out this calculation. To be precise, one should include the Gibbons-Hawking-York boundary term in the gravitational action (1.1) in addition to the scalar curvature. Then, after suitably modifying the Gauss-Bonnet argument in (1.2), one should emend (1.3) to include a sum over the number of boundary components, as well as an integral over the corresponding boundary lengths. These changes have the effect of integrating over *all* of the moduli available to a 2-dimensional Euclidean manifold. I would be curious to learn whether such a calculation exists in the literature.

More interesting than the partition function Z is its extension to the generating functional $Z[J]$ of correlation functions of a given theory. It seems rather complicated to directly evaluate a version of (1.1) with arbitrary sources J added, not least because it is not obvious what the observables in a topological theory of gravity should be. Witten's answer—that the correlators of 2D gravity are intersection numbers on the moduli spaces discussed above—puts the theory on solid footing. In the process developing intersection theory on $\overline{\mathcal{M}}_{g,n}$ using symplectic reduction and the Duistermaat–Heckman theorem, we are led to define the cohomology classes ψ of forms that live on $\overline{\mathcal{M}}_{g,n}$. Their integrals over $\overline{\mathcal{M}}_{g,n}$ constitute the aforementioned correlation functions, and they can be used to construct the extremely useful MMM classes κ. One of the κ classes is unexpectedly cohomologous to a multiple of the WP symplectic form, and Witten was able to use this relation to derive a recursive formula for the correlators in terms of the WP volumes themselves. The upshot is that the volumes $V_{g,n}$ contain a great deal more information than initially meets the eye.

So by carefully studying the moduli spaces of Riemann surfaces, we are able to do calculations in and effectively solve the quantum-gravitational theory of these surfaces.

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